

THE ABEL–JACOBI MAP OF A MODULI COMPONENT OF VECTOR BUNDLES ON THE CUBIC THREEFOLD

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ABSTRACT. The Abel–Jacobi map of the family of elliptic quintics lying on a general cubic threefold is studied. It is proved that it factors through a moduli component of stable rank 2 vector bundles on the cubic threefold with Chern numbers $c_1 = 0, c_2 = 2$, whose general point represents a vector bundle obtained by Serre’s construction from an elliptic quintic. The elliptic quintics mapped to a point of the moduli space vary in a 5-dimensional projective space inside the Hilbert scheme of curves, and the map from the moduli space to the intermediate Jacobian is étale. As auxiliary results, the irreducibility of families of elliptic normal quintics and of rational normal quartics on a general cubic threefold is proved. This implies the uniqueness of the moduli component under consideration. The techniques of Clemens–Griffiths and Welters are used for the calculation of the infinitesimal Abel–Jacobi map.

INTRODUCTION

In their famous paper [Cl-Gr], Clemens and Griffiths represented the intermediate Jacobian $J_1(X)$ of a smooth cubic threefold X as the Albanese variety of the family of lines on X and parametrized the theta divisor on it by the Abel–Jacobi images of pairs of lines. They used this parametrization to establish the Torelli Theorem for X and the non-rationality of X . Similar results were obtained by Tyurin [Tyu]. Later several authors constructed parametrizations of the intermediate Jacobians and, sometimes, of their theta divisors for other Fano threefolds, using families of curves of low degree lying on them, see, for example, [Ce-Ve], [De-1], [De-2], [I], [Lo], [Le], [T-1], [T-2], [V-1], [V-2], [We].

However, the following natural question has not been investigated: given a family of connected curves parametrized by some variety B , what are the fibers of the Abel–Jacobi map considered on B itself and not on the $\text{Alb}(B)$? By analogy with the Abel–Jacobi map from

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divisors of a given degree d on a curve of genus g to its Jacobian, which is smooth with projective spaces as fibers, provided that $d \geq 2g - 1$, we might expect that the Abel–Jacobi map in our context behaves well for families of curves of rather big degree. The papers mentioned above search, on the contrary, for the families of the smallest possible degree whose image generates the intermediate Jacobian. For the cubic threefold X , the only known facts in this direction concern the map of difference, defined on the pairs of lines in X , which is onto the theta divisor and is generically of degree 6 (see 1.7), and the Abel–Jacobi map of the family of rational cubics in X , which is also onto the theta divisor and its generic fiber is \mathbb{P}^2 (see Proposition 3.2). There are no similar results about the Abel–Jacobi parametrizations of the entire intermediate Jacobian.

In the present paper, we study the Abel–Jacobi map on the 10-dimensional component H of the Hilbert scheme whose general point represents an elliptic normal quintic in X . We find an open subset $\mathcal{H} \subset H$, on which the map is smooth and all the components of its fibers are isomorphic to \mathbb{P}^5 (Theorem 5.6). The analogy with the Jacobians of curves is not complete, because the fibers may consist of *many* copies of the projective space, and almost nothing is known about the boundary of \mathcal{H} in H (see Proposition 5.7).

We give also an unexpected interpretation to the variety $M = M_X$ parametrizing the connected components of the fibers of the Abel–Jacobi map $\Phi : \mathcal{H} \rightarrow J_1(X)$: we show that it is isomorphic to an open subset in the component of the moduli space $M_X(2; 0, 2)$ of stable rank 2 vector bundles on X with Chern numbers $c_1 = 0, c_2 = 2$, whose general member is obtained by Serre’s construction from elliptic quintics in X . Thus, there exists a natural Abel–Jacobi map from this component of the moduli space of stable vector bundles on X to $J_1(X)$, induced by Φ ; this map is nothing but the Abel–Jacobi map of the moduli space, defined by the second Chern class of the vector bundle with values in the Chow group of 1-cycles modulo rational equivalence. Moreover, this map is étale on the open subset M , see Theorem 5.6.

As far as we know, the above theorem provides the first example of a moduli space of vector bundles which has a dominant map to an abelian variety, different from the Picard and Albanese varieties of the base. It also shows that the moduli spaces of vector bundles yield sometimes more efficient parametrizations of the intermediate Jacobian of a threefold than families of curves.

This construction can be relativized over the family of nonsingular hyperplane sections of a cubic fourfold V . It gives a moduli component of torsion sheaves on V with supports on the hyperplane sections

$X = H \cap V$, whose restrictions to X are rank two vector bundles from M_X . One can show that the Yoneda pairing induces a symplectic structure on it. This is a new example of a moduli space of sheaves possessing a symplectic structure; the known ones parametrize sheaves on hyperkaehler manifolds (see [Muk] for surfaces, and [K] for higher dimensional varieties). It is a 10-dimensional symplectic variety, covering the one constructed by Donagi and Markman in [D-M, Example 8.22] (work in process).

Now, we will briefly describe the contents of the paper by sections.

In Section 1, we remind basic facts about the cubic threefolds, the Abel–Jacobi map and its differential, following essentially [Cl-Gr] and [We].

Section 2 explains Serre’s construction, applied to elliptic normal quintics in X . It yields a 5-dimensional component of the moduli space of stable vector bundles of rank 2 on X . The unstability of vector bundles obtained by Serre’s construction from *non-normal* elliptic quintics is also proved.

Section 3 gives the proof of the irreducibility of the family of rational normal quartics $\Gamma \subset X$. First, the irreducibility of the family of rational twisted cubics lying in nonsingular hyperplane sections of X is proved, in using the Abel–Jacobi map of these curves to the theta divisor in $J_1(X)$. Next, the irreducibility of the family of curves of the form $D + l$ is proved, where D is a twisted cubic as above and l a line meeting D transversely at 1 point. We show also that the Hilbert scheme Hilb_X^{4n+1} is smooth at points representing such curves $D + l$, and that they are strongly smoothable into a rational normal quartic (the techniques of [H-H] are used). This implies the wanted result.

Section 4 is devoted to the normal elliptic quintics $C \subset \mathbb{P}^4$ contained in a cubic threefold X . It is proved that the Hilbert scheme Hilb_X^{5n} is smooth at points representing such curves C , and that their family is irreducible for a *general* X . The proof is reduced to that of the irreducibility of the family of rational normal quartics $\Gamma \subset X$, in using the liaison defined by cubic scrolls $\mathbb{F}_1 \subset \mathbb{P}^4$: $\mathbb{F}_1 \cap X = C \cup \Gamma$.

Section 5 contains the main result of the paper. We relativize Serre’s construction of Section 2 over some open subset $H_0 \subset H$, and construct a morphism $\phi : H_0 \rightarrow M_X(2; 0, 2)$ with fibers \mathbb{P}^5 . We prove that ϕ is, locally in the étale topology, the structure projection of a projectivized vector bundle of rank 6. The smoothness of the Abel–Jacobi map $\Phi : \mathcal{H} \rightarrow J_1(X)$ is proved in using the technique of Tangent Bundle Sequence, and this, together with the fact that an Abel–Jacobi map should contract to a point any projective space, implies that the components of fibers of Φ are exactly the fibers of ϕ , isomorphic to \mathbb{P}^5 . We

also study the extension $\tilde{\Phi}$ of Φ to the boundary component of $\mathcal{H} \subset H$, formed by non-normal elliptic quintics, and find that its image coincides with a translate of $F + F \subset \text{Alb}(F)$, where F is the Fano surface of X , and that the differential of $\tilde{\Phi}$ is degenerate along this component.

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1. GENERALITIES AND KNOWN RESULTS

1.1. We will start by a reminder on cubic threefolds. Let X be a smooth cubic hypersurface in \mathbb{P}^4 . It is a Fano variety, that is its anticanonical sheaf $\omega_X^{-1} \simeq \mathcal{O}_X(2)$ is ample, and the following properties are obtained by standard techniques for Fano threefolds [Isk-1]:

$$\begin{aligned} h^i(\mathcal{O}_X(k)) &= 0 \text{ for } i = 1, 2, k \in \mathbb{Z}, \\ h^{i,0} = h^{0,i} &= 0 \text{ for } i > 0, h^{1,2} = h^{2,1} = 5, \end{aligned} \quad (1)$$

$$\begin{aligned} \text{Pic}(X) = A_2(X) = H^2(X, \mathbb{Z}) &= \mathbb{Z} \cdot [\mathcal{O}_X(1)], \\ B_1(X) = H^4(X, \mathbb{Z}) &= \mathbb{Z} \cdot l. \end{aligned} \quad (2)$$

Following [Fu, Ch. 19], we denote by A_i , resp. B_i the Chow group of algebraic cycles of dimension i modulo rational, resp. algebraic equivalence, and l is the class of a line.

1.2. The geometry of lines (and, sometimes, of conics) plays a crucial role in the study of a Fano threefold X , essentially by the following two reasons. Firstly, they give (a part of) generators of the group of birational automorphisms of X [Isk-2], and secondly, they are used for the description of the Abel–Jacobi map $AJ : \text{Hom}_1 X \rightarrow J_1(X)$, where we denote, following again [Fu], by $\text{Hom}_i X$ the subgroup in the group $Z_i X$ of algebraic i -cycles, consisting of cycles homologous to 0, and $J_1(X)$ is the intermediate Jacobian of X . According to [Cl-Gr], the lines on a nonsingular cubic threefold X are parametrized by a nonsingular surface $F = F(X) \subset \text{Gr}(2, 5)$ with invariants

$$h^{1,0}(F) = 5, \quad h^{2,0}(F) = 10, \quad (3)$$

which can be thought of as a component of the Hilbert scheme Hilb_X^{n+1} . The smoothness of Hilb_X^{n+1} at the points of F follows from the calculation of the normal bundle of a line $l \subset X$:

$$\begin{aligned}\mathcal{N}_{l/X} &\simeq \mathcal{O} \oplus \mathcal{O} \quad (\text{for } l \in F \setminus D) \\ \mathcal{N}_{l/X} &\simeq \mathcal{O}(-1) \oplus \mathcal{O}(1) \quad (\text{for } l \in D),\end{aligned}\tag{4}$$

where $D \subset F$ is a curve. Formulas (4) imply that $h^1(\mathcal{N}_{l/X}) = 0$, hence F is smooth by [Gro]. F is called the *Fano surface* of X .

1.3. We will recall some known facts about the Abel–Jacobi map [Gri], [Cl-Gr], [Tyu], [We]. The intermediate Jacobian of a threefold X is defined by

$$J_1(X) = (F^2 H^3(X, \mathbb{C}))^* / \text{im}(H_3(X, \mathbb{Z})),$$

where $F^2 = H^{3,0} + H^{2,1}$ is a term of the Hodge filtration, and $H_3(X, \mathbb{Z})$ is mapped to $(F^2 H^3(X, \mathbb{C}))^*$ by integration over cycles. For a cubic threefold, $H^{3,0}(X) = 0$, so, by the Hodge index theorem, the intersection form on $H_3(X)$ defines a principal polarization on $J_1(X)$.

The Abel–Jacobi map is defined as follows: let $\gamma \in \text{Hom}_1 X$. Then there exists $\Gamma \in H_3(X, \mathbb{Z})$ such that $\gamma = \partial\Gamma$, and $AJ(\gamma)$ is given by

$$\left[\omega \mapsto \int_{\Gamma} \omega \right] \quad \text{mod } \text{im } H_3(X, \mathbb{Z}).$$

For a family of 1-cycles $\{Z_b\}$ parametrized by some variety B with a reference point β , this defines a set-theoretic map $AJ_B : B \rightarrow J_1(X)$, $b \mapsto AJ(Z_b - Z_{\beta})$. By [Gri, II], AJ_B is analytic if B is nonsingular. Hence AJ_B factors through the Albanese variety of (the desingularization of) B ; the resulting morphism $\text{Alb}(B) \rightarrow J_1(X)$ is also called Abel–Jacobi map. Applying this construction to the family of lines on a cubic threefold X , Clemens–Griffiths get a natural map $\text{Alb}(F) \rightarrow J_1(X)$. By (1) and (3), both abelian varieties are of dimension 5. The following facts are known:

1.4. The Abel–Jacobi map establishes an isomorphism of abelian varieties $\text{Alb}(F) \simeq J_1(X)$ ([Cl-Gr, (0.9)]).

1.5. For any fixed $s \in F$, the map $i_s : F \rightarrow J_1(X)$, $t \mapsto [l_t - l_s]$ is a closed embedding, where l_t denotes the line in X corresponding to a point $t \in F$ ([Tyu, §3]).

1.6. The class of $i_s(F)$ in $B_2(J_2(X))$ is $\frac{1}{3!}\theta^3$, where θ is the class of the theta divisor defining the principal polarization on $J_2(X)$ ([Cl-Gr, (0.9)]).

1.7. The map $\psi : F \times F \longrightarrow J_1(X)$, $(s, t) \mapsto [l_t - l_s]$ is generically 6-to-1 over the image, and $[\psi(F \times F)] = \theta$ (ibid, (0.10)).

A precise description of the singularities of the map ψ is given in [Tyu]. See also loc. cit., or [Fu, Example 4.3.2] and references therein for further information on the geometry of the Abel–Jacobi map of lines.

1.8. We will use in the sequel another map $\tilde{\psi} : F \times F \longrightarrow J_1(X)$, $(s, t) \mapsto [l_s + l_t - 2l_0]$, where $0 \in F$ is some reference point. The same arguments as in [Cl-Gr] show that $\tilde{\psi}$ is generically finite over its image and 1.6 implies that its class in $B_2(J_1(X))$ is $m\theta$ for some $m \in \mathbb{Z}$, $m > 0$. The generical finiteness follows from the tangent bundle theorem for F [Cl-Gr, 12.4], which implies that whenever l_s, l_t is a pair of skew lines, the tangent spaces to $i_0(F)$ in $\text{Alb}(F)$ at $i_0(s), i_0(t)$ are transversal (when translated to $0 \in \text{Alb}(F)$), so that both maps of sum ($\tilde{\psi}$) and of difference (ψ) have injective differentials at (s, t) . Beauville communicated us that the values of the degree d of the map $\tilde{\psi} : F \times F \longrightarrow \tilde{\psi}(F \times F)$ and of m follow from the description of the intermediate Jacobian in terms of Prym varieties [B]: $d = 2$ and $m = 3$.

1.9. **Technique of TBS.** Now we will describe, following [We, Sect. 2], the technique of “tangent bundle sequence”, which will be applied later to elliptic quintics in a cubic threefold.

Let $X \hookrightarrow W$ be an embedding of a smooth projective threefold X into a smooth quasiprojective fourfold W , and $\{Z_b\}_{b \in B}$ a flat family of curves in X parametrized by a nonsingular variety B over \mathbb{C} . Then a choice of a base point $\beta \in B$ determines the Abel–Jacobi map $B \longrightarrow J_1(X)$. Let $Z = Z_b$ be a scheme theoretical locally complete intersection fiber of the family. Then the differential of the Abel–Jacobi map at b factors into the composition of the two natural maps:

$$T_{B,b} \longrightarrow H^0(Z, \mathcal{N}_{Z/X})$$

and

$$\psi_Z : H^0(Z, \mathcal{N}_{Z/X}) \longrightarrow (H^0(X, \Omega_X^3) \oplus H^1(X, \Omega_X^2))^* = T_{J_1(X), 0}$$

The $H^0(X, \Omega_X^3)^*$ -component of ψ_Z is always 0, so in the sequel we will consider and denote by the same symbol ψ_Z the map to $H^1(X, \Omega_X^2)^*$. In [We], this factorization is given for Z smooth with a reference to [Gri, II]. This holds also in a more general context [Fl]: if we identify $H^0(\mathcal{N}_{Z/X})$ with $\text{Ext}_X^1(\mathcal{F}, \mathcal{F})$, where $\mathcal{F} = \mathcal{O}_Z$, then ψ_Z can be represented as the composition $\text{Ext}_X^1(\mathcal{F}, \mathcal{F}) \xrightarrow{\cup \text{at}(\mathcal{F})} \text{Ext}_X^2(\mathcal{F}, \mathcal{F} \otimes \Omega^1) \xrightarrow{\text{Tr}} H^2(\Omega^1)$, where $\text{at}(\mathcal{F}) \in \text{Ext}_X^1(\mathcal{F}, \mathcal{F} \otimes \Omega^1)$ denotes the Atiyah–Illusie class of a coherent sheaf \mathcal{F} .

It is convenient to describe ψ_Z via its conjugate ψ_Z^* , which fits into the following commutative square:

$$\begin{array}{ccc} H^0(X, \mathcal{N}_{X/W} \otimes \omega_X) & \xrightarrow{R} & H^1(X, \Omega_X^2) \\ r_Z \downarrow & & \downarrow \psi_Z^* \\ H^0(Z, \mathcal{N}_{X/W} \otimes \omega_X|_Z) & \xrightarrow{\beta_Z} & H^0(Z, \mathcal{N}_{Z/X})^*. \end{array} \quad (5)$$

Here r_Z is the map of restriction to Z , and the whole square (upon natural identifications) is a part of the commutative diagram of long exact cohomology sequences associated to the following commutative diagram of sheaves:

$$\begin{array}{ccccccccc} 0 & \rightarrow & \Omega_X^2 \otimes \mathcal{O}_Z & \rightarrow & \Omega_W^3 \otimes \mathcal{N}_{X/W} \otimes \mathcal{O}_Z & \rightarrow & \Omega_X^3 \otimes \mathcal{N}_{X/W} \otimes \mathcal{O}_Z & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \rightarrow & \Omega_X^3 \otimes \mathcal{N}_{Z/X} & \rightarrow & \Omega_X^3 \otimes \mathcal{N}_{Z/W} & \rightarrow & \Omega_X^3 \otimes \mathcal{N}_{X/W} \otimes \mathcal{O}_Z & \rightarrow & 0 \end{array} \quad (6)$$

See [We, 2.8] for definitions of the maps.

2. VECTOR BUNDLES FROM ELLIPTIC QUINTICS

In the sequel, X will denote a nonsingular cubic threefold in \mathbb{P}^4 , H or $H(X)$ the union of components of Hilb_X^{5n} having smooth normal elliptic quintics $C \in \mathbb{P}^4$ as their generic points. We will see later in Section 4 that $H(X)$ is non empty and has dimension 10 for any X . We will denote by $[C]_V$ the point representing a subscheme $C \subset V$ in the Hilbert scheme of V ; the subscript V may be omitted, if this makes no confusion.

In this section, we will study the vector bundles \mathcal{E} on X obtained by Serre's construction applied to an elliptic quintic C in X . They are defined by the following extension of \mathcal{O}_X -modules:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(1) \longrightarrow \mathcal{I}_C(2) \longrightarrow 0, \quad (7)$$

where $\mathcal{I}_C = \mathcal{I}_{C,X}$ is the ideal sheaf of C in X . Since the class of C in $B_1(X)$ is $5l$, the sequence (7) implies that $c_1(\mathcal{E}) = 0, c_2(\mathcal{E}) = 2l$. Moreover, $\det \mathcal{E}$ is trivial, and hence \mathcal{E} is self-dual as soon as it is a vector bundle (that is, $\mathcal{E}^* \simeq \mathcal{E}$).

We will verify that there exists a unique non-trivial extension (7) for given C (Corollary 2.2).

Lemma 2.1. *For any smooth $C \in H$, we have:*

- a) $\dim \text{Ext}^1(\mathcal{I}_C(2), \mathcal{O}_X) = 1$;
- b) for $k \geq 2$, $h^0(\mathcal{I}_C(k)) = \binom{4+k}{4} - \binom{1+k}{4} - 5k$, $h^1(\mathcal{I}_C(k)) = h^2(\mathcal{I}_C(k)) = 0$; $h^0(\mathcal{I}_C(1)) = h^1(\mathcal{I}_C(1))$ is 0 if C is not contained in a hyperplane, and 1 otherwise, and $h^2(\mathcal{I}_C(1)) = 0$;

c) $h^0(\mathcal{I}_C(2)) = 5, h^0(\mathcal{E}(1)) = 6, h^i(\mathcal{E}(1)) = 0$ for $i > 0$.

Proof. a) Applying $\text{Ext}(\cdot, \mathcal{O}_X)$ to the restriction sequence

$$0 \longrightarrow \mathcal{I}_C(k) \longrightarrow \mathcal{O}_X(k) \longrightarrow \mathcal{O}_C(k) \longrightarrow 0 \quad (8)$$

for $k = 2$, we obtain:

$$\begin{aligned} \text{Ext}^1(\mathcal{O}_X(2), \mathcal{O}_X) &\longrightarrow \text{Ext}^1(\mathcal{I}_C(2), \mathcal{O}_X) \longrightarrow \text{Ext}^2(\mathcal{O}_C(2), \mathcal{O}_X) \\ &\longrightarrow \text{Ext}^2(\mathcal{O}_X(2), \mathcal{O}_X) \end{aligned} \quad (9)$$

By (1), the left and the right hand terms are zero. So, $\text{Ext}^1(\mathcal{I}_C(2), \mathcal{O}_X) = \text{Ext}^2(\mathcal{O}_C(2), \mathcal{O}_X) = H^0(X, \mathcal{E}xt_{\mathcal{O}_X}^2(\mathcal{O}_C, \omega_X)) = H^0(C, \omega_C) \simeq \mathbb{C}$.

b) Looking again at (8), we see that the assertion for $k \geq 1$ follows from the projective normality of C [Hu, Proposition IV.1.2] in the case when C does not lie in a hyperplane.

Let now $C \subset \mathbb{P}^3$. If $k = 1$, the assertions follow easily from (8). For $k = 2$, it is sufficient to show that C is not contained in a quadric. Assume the contrary. C is obviously not contained in a nonsingular quadric, neither in the nonsingular locus of a quadratic cone, because in that case its degree should be even. So it lies in a quadratic cone $Q \subset \mathbb{P}^3$ and passes through the vertex. By degree considerations, it has to meet twice the generators of the cone outside the vertex. So, in blowing up the vertex, we obtain a nonsingular curve \tilde{C} of genus 1 in the Hirzebruch surface \mathbb{F}_2 with class $2s + af$, where s, f are the standard generators of $\text{Pic } \mathbb{F}_2$ such that $s^2 = -2, f^2 = 0, (s \cdot f) = 1$. Computing the canonical class of \tilde{C} , we find immediately $a = 4$, hence $(s \cdot \tilde{C}) = 0$, which implies that C does not pass through the vertex. This contradicts our assumptions. Hence C does not lie in any quadric.

We have seen that $h^1(\mathcal{I}_C(2)) = h^2(\mathcal{I}_C(1)) = 0$; obviously, also $h^3(\mathcal{I}_C) = 0$, so that C is 3-regular according to the definition of Castelnuovo–Mumford. The Castelnuovo’s Proposition in Lecture 14 of [Mum-1] implies the case $k > 2$.

c) The first equality follows from b) with $k = 2$. The remaining ones are immediate consequences of (7) and of the Serre duality. \square

Corollary 2.2. *For any smooth $C \in H$, there exists a unique (up to isomorphism) extension (7) with \mathcal{E} locally free.*

Proof. The uniqueness follows from Lemma 2.1, a). The local freeness follows from the sheafified version of (9), giving $\mathcal{E}xt_{\mathcal{O}_X}^1(\mathcal{I}_C(2), \mathcal{O}_X) = \mathcal{E}xt_{\mathcal{O}_X}^2(\mathcal{O}_C, \omega_X) = \omega_C$, and from the following lemma due to Serre:

Lemma 2.3 ([OSS, Ch. I, Lemma 5.1.2]). *Let X be a nonsingular variety, C a locally complete intersection of codimension 2 in X , L invertible on X , and*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_C \otimes L \longrightarrow 0$$

an extension given by a class $e \in \text{Ext}^1(\mathcal{I}_C \otimes L, \mathcal{O}_X)$. Then \mathcal{E} is locally free of rank 2 if and only if the image of e in $H^0(C, \text{Ext}_{\mathcal{O}_X}^1(\mathcal{I}_C \otimes L, \mathcal{O}_X))$ generates the stalk of $\text{Ext}_{\mathcal{O}_X}^1(\mathcal{I}_C \otimes L, \mathcal{O}_X)$ at every point of C .

□

Corollary 2.4. *If C is smooth and is not contained in a hyperplane, then any non-zero section of $\mathcal{E}(1)$ has a locally complete intersection curve with zero canonical class as its zero locus, and this defines a natural morphism $\mathbb{P}^5 = \mathbb{P}(H^0(X, \mathcal{E}(1))) \longrightarrow H$ whose image contains $[C]$.*

Proof. The scheme of zeros of a section of a rank 2 vector bundle on a nonsingular variety is locally complete intersection as soon as it is of codimension 2; the assertion about the canonical class follows by adjunction. So, we have to show that a non-zero section of $\mathcal{E}(1)$ cannot vanish on a surface $S \subset X$. By (2), $\mathcal{O}_X(S) \simeq \mathcal{O}_X(d)$ for some $d > 0$, hence it suffices to show that $h^0(\mathcal{E}(1-d)) = 0$ for all $d > 0$. Twisting (7) by $\mathcal{O}_X(-1)$, we see that if C is not contained in a hyperplane, that is, $h^0(\mathcal{I}_C(1)) = 0$, then $h^0(\mathcal{E}) = 0$, and so much the more $h^0(\mathcal{E}(1-d)) = 0$. This ends the proof. □

For future use in Section 5, we will study also the case of non-normal elliptic quintics.

Proposition 2.5. *If C is a smooth elliptic quintic contained in a hyperplane section S of X , then there exists a unique pair of possibly coincident lines $l_1 \cup l_2$ in S , such that the zero locus of any section of $\mathcal{E}(1)$, if not of codimension 2, is of the form $S' \cup l_1 \cup l_2$ for some hyperplane section S' of X . In this case C is rationally equivalent to a curve of the form $C^3 + l_1 + l_2$, where C^3 is a plane cubic in S .*

Conversely, any curve of the form $C^3 + l_1 + l_2$ on a general cubic threefold X , where C^3 is a plane cubic curve and l_1, l_2 a pair of disjoint lines, is rationally equivalent to a smooth elliptic quintic contained in a hyperplane section of X .

Proof. As in the previous corollary, we find a section s of $\mathcal{E}(1)$ vanishing on S , and a section s_0 of \mathcal{E} such that $s = Ls_0$, where L is the linear form defining S in X . As $c_2(\mathcal{E}) = 2l$, the scheme of zeros Z of s_0 is either a conic, or a pair of lines; Z may be non-reduced (a double line),

but without embedded points, for it is a locally complete intersection. As $h^0(\mathcal{E}(1)) = 6$, we deduce from the exact triple

$$0 \longrightarrow \mathcal{O}_X(1) \xrightarrow{\alpha} \mathcal{E}(1) \xrightarrow{\beta} \mathcal{I}_Z(1) \longrightarrow 0 \quad (10)$$

that $h^0(\mathcal{I}_Z(1)) = 1$, so that Z is contained in a unique hyperplane. This eliminates the case of the conic, as well as that of a planar double line. Thus, $\chi(\mathcal{O}_Z(n)) = 2n + 2$, $\langle Z \rangle = P^3$, i. e. either Z is a disjoint union $Z = l_1 \sqcup l_2$ of two lines, or Z is a double non-planar locally complete intersection structure on a line $l_1 = l_2$ (the latter is clearly the limit of the former). We will denote this shortly as $Z = l_1 \cup l_2$. Moreover, we see from (10) that $\mathcal{E}(1)$ has a 5-dimensional subspace of sections $L's_0$ with $L' \in H^0(X, \mathcal{O}_X(1))$ whose zero locus is of the form $S' \cup l_1 \cup l_2$, where $S' = \{L' = 0\}$ is a hyperplane section of X .

Let s_1 be the section of $\mathcal{E}(1)$ vanishing exactly on C , and L' a general linear form. Then the pencil of sections $\lambda_0 L's_0 + \lambda_1 s_1$ defines a pencil of curves, degenerating C into $C^3 + l_1 + l_2$ with $C^3 \subset S'$ of degree 3.

Now, s_1 is not in the image of α in (10), and it vanishes on C . Hence $\beta(s_1) \neq 0$ and $\beta(s_1)$ vanishes on C . Hence C lies in the unique hyperplane containing $Z = l_1 \cup l_2$. Hence, this hyperplane is $L = 0$, and $l_1 \cup l_2 \subset S$.

It remains to see that $C^3 \subset S$. By construction, $C^3 \subset S'$ for another hyperplane section S' of X . If C^3 were not contained in $S \cap S'$, then the general member of the pencil of curves defined by the sections $\lambda_0 L's_0 + \lambda_1 s_1$ would be a smooth elliptic curve not contained in a hyperplane, because $C^3 \cup l_1 \cup l_2$ is not. This contradicts Corollary 2.4. Hence $C^3 \subset S \cap S'$ is a plane cubic curve.

The fact that any curve of the form $C^3 + l_1 + l_2$ with $l_1 \cap l_2 = \emptyset$ is rationally equivalent to a smooth quintic is reduced to the study of linear systems on cubic surfaces. We can always arrange the things so that C^3, l_1, l_2 lie in one hyperplane and C^3 meets each one of the lines l_1, l_2 transversely at 1 point, because all the plane sections C^3 of X are rationally equivalent to each other, being parametrized by the rational Grassmannian variety $G(3, 5)$. We obtain in this way the linear system $|C^3 + l_1 + l_2|$ on the cubic surface $S = \langle l_1, l_2 \rangle \cap X$. If S is nonsingular, we can represent it as \mathbb{P}^2 with 6 points P_1, \dots, P_6 blown up. Let e_0 be the inverse image of a line in \mathbb{P}^2 , and e_i the exceptional divisors over P_i ($i = 1, \dots, 6$). We can choose this representation in such a way, that $l_1 = e_1, l_2 = e_2$. We have $C^3 \in |h|$, where $h = 3e_0 - e_1 - \dots - e_6$ is the hyperplane section, so that $C^3 + l_1 + l_2 \in |3e_0 - e_3 - \dots - e_6|$ is obviously smoothable in its linear system.

To treat the case when S is singular, we will use the assumption that X is general. This restricts the possible degenerations of the hyperplane

sections $H \cap X$ to those which have codimension ≤ 4 in the projective space \mathbb{P}^{19} parametrizing cubic surfaces in \mathbb{P}^3 . Such degenerations S have at most 4 isolated singular points, and the line joining two distinct singular points lies entirely in S (see [Gr-Ha, Sect. 4.6]). Choosing coordinates in such a way that the singular points lie in the vertices of the coordinate octahedron and writing out the monomials that can be present in the equation of S in order that S might have isolated singularities in the prescribed vertices, we arrive at the following list of degenerations up to codimension 4: $A_1, A_1A_1, A_2, A_1A_1A_1, A_2A_1, A_3, A_1A_1A_1A_1, A_2A_1A_1, A_3A_1, A_2A_2, D_4$, where X_i (resp. X_iX_j, \dots) stands for a surface with one singular point of type X_i (resp. distinct singular points of types X_i, X_j, \dots), and the analytic types of singularities X_i are either A_i or D_i of Du Val, defined up to analytic equivalence by $x^2 + y^2 + z^{i+1} = 0$ (A_i) or $x^2 + y^2z + z^{i-1} = 0$ (D_i). The codimension of such degenerations in \mathbb{P}^{19} is i (resp. $i + j + \dots$).

The standard technique of projections from two lines (see [M] or [Re]) gives a birational map $S \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. In the above 11 cases, it represents S as $\mathbb{P}^1 \times \mathbb{P}^1$ with 5 points $\Psi = \{Q_1, \dots, Q_5\}$ blown up and (-2) -curves blown down; some of the points Q_i can be infinitesimal. Since the quadric with a point blown up is isomorphic to \mathbb{P}^2 with two points blown up, we can replace $(\mathbb{P}^1 \times \mathbb{P}^1, \Psi)$ by (\mathbb{P}^2, Φ) , where $\Phi = \{P_1, \dots, P_6\}$ is a set of 6 points, some of which can be infinitesimal. Let S_Φ denote the blowup of Φ in \mathbb{P}^2 .

We are going to provide a configuration of Φ giving rise to each of the 11 degenerations above. In each case, Φ should be general satisfying the formulated constraints in the sense that there are no curves $F \simeq \mathbb{P}^1, F^2 \leq -2$ among the proper transforms in S_Φ of lines or conics in \mathbb{P}^2 except for the (-2) -curves imposed by the constraints. Here is the list:

- A_1 : P_1, P_2, P_3 are collinear.
- A_1A_1 : $P_3 = P_1P_2 \cap P_4P_5$.
- A_2 : P_1, P_2, P_3 are collinear, and P_4, P_5, P_6 are collinear.
- $A_1A_1A_1$: P_2, P_4, P_6 lie on the sides of the triangle $P_1P_3P_5$.
- A_2A_1 : The limit of A_1A_1 when $P_4 \rightarrow P_2$ along the line P_2P_5 (so that P_4 is infinitesimally close to P_2).
- A_3 : $P_3 = P_1P_2 \cap P_4P_5$, and P_6 is infinitesimally close to P_3 .
- $A_1A_1A_1A_1$: Φ is the set of intersection points of 4 lines in general position in \mathbb{P}^2 .
- $A_2A_1A_1$: The specialization of A_2A_1 with $P_6 \in P_1P_5$.
- A_3A_1 : The limit of A_2A_1 when $P_6 \rightarrow P_1$.
- A_2A_2 : The limit of A_2A_1 when $P_6 \rightarrow P_5$.

D_4 : P_1, P_2, P_3 are collinear, and P_{3+i} is infinitesimally close to P_i ($i = 1, 2, 3$).

A routine verification of the emptiness of the base loci in S_Φ of the linear systems $h + l_1 + l_2$ ends the proof. \square

Proposition 2.6. *Let C be a smooth elliptic quintic. If C is not contained in a hyperplane, then the vector bundle \mathcal{E} obtained from C by Serre's construction is stable. It has a non-trivial global section and is Gieseker unstable, if C is contained in a hyperplane.*

Proof. Unstability part. Assume that C is contained in a hyperplane. As in the proof of Proposition 2.5, we obtain an exact triple (10). It is obvious that $\chi(\mathcal{O}_X(k)) > \chi(\mathcal{I}_Z(k))$ for $k \gg 0$, so \mathcal{E} is Gieseker unstable, though semistable in the sense of Mumford–Takemoto.

Stability part. Any saturated torsion free rank 1 subsheaf of $\mathcal{E}(1)$ is invertible of the form $\mathcal{O}_X(k)$ and gives an exact triple:

$$0 \longrightarrow \mathcal{O}_X(k) \xrightarrow{\alpha} \mathcal{E}(1) \longrightarrow \mathcal{I}_Z(2-k) \longrightarrow 0 \quad (11)$$

where $Z \subset X$ is a subscheme of codimension 2 (if not empty). Clearly, (7) twisted by $\mathcal{O}_X(-2)$ shows that the case $k \geq 2$ here is impossible. On the other hand, if \mathcal{E} is not Gieseker stable, then, computing Hilbert polynomials, one has $k \geq 1$ in (11). Hence $k = 1$, and we obtain the above case when C lies in a hyperplane. \square

Lemma 2.7. *Let C be a smooth elliptic quintic in X . Let \mathcal{E} be the unique locally free sheaf defined by (7). Then we have:*

- a) $h^i(\mathcal{E}(-1)) = 0 \ \forall i \in \mathbb{Z}$;
- b) $h^0(\mathcal{E} \otimes \mathcal{E}) = 1$ if and only if C is not contained in a hyperplane, and in this case $h^1(\mathcal{E} \otimes \mathcal{E}) = 5, h^2(\mathcal{E} \otimes \mathcal{E}) = h^3(\mathcal{E} \otimes \mathcal{E}) = 0$;
- c) $h^0(\mathcal{N}_{C/X}) = 10, h^1(\mathcal{N}_{C/X}) = 0$.

Proof. a) From (8) with $k = 0$, we deduce $h^i(\mathcal{I}_C) = 0, i = 0, 1, 3, h^2(\mathcal{I}_C) = 1$. Then (7), twisted by $\mathcal{O}_X(-2)$, gives $\chi(\mathcal{E}(-1)) = h^i(\mathcal{E}(-1)) = 0$ for $i = 0, 1$. Next, by Serre duality, $h^3(\mathcal{E}(-1)) = h^0(\mathcal{E}(-1)) = 0$; hence also $h^2(\mathcal{E}(-1)) = 0$.

b) Tensor (7) by $\mathcal{E}(-1)$:

$$0 \longrightarrow \mathcal{E}(-1) \longrightarrow \mathcal{E} \otimes \mathcal{E} \longrightarrow \mathcal{I}_C \otimes \mathcal{E}(1) \longrightarrow 0 \quad (12)$$

From a) and (12), we deduce that

$$h^i(\mathcal{E} \otimes \mathcal{E}) = h^i(\mathcal{I}_C \otimes \mathcal{E}(1)) \ \forall i \in \mathbb{Z}. \quad (13)$$

If C is not contained in a hyperplane, then \mathcal{E} is stable, hence by [OSS, Theorem I.2.9], \mathcal{E} is simple, that is $h^0(\mathcal{H}om(\mathcal{E}, \mathcal{E})) = h^0(\mathcal{E} \otimes \mathcal{E}) = 1$.

By Serre duality, $h^3(\mathcal{E} \otimes \mathcal{E}) = h^0(\mathcal{E} \otimes \mathcal{E}(-2)) = 0$. By Riemann–Roch–Hirzebruch, $\chi(\mathcal{E} \otimes \mathcal{E}) = -4$. It remains to prove that $h^2(\mathcal{E} \otimes \mathcal{E}) = h^1(\mathcal{E} \otimes \mathcal{E}(-2)) = 0$.

Consider the exact triples

$$0 \longrightarrow \mathcal{E}(-3) \longrightarrow \mathcal{E} \otimes \mathcal{E}(-2) \longrightarrow \mathcal{I}_C \otimes \mathcal{E}(-1) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{I}_C \otimes \mathcal{E}(-1) \longrightarrow \mathcal{E}(-1) \longrightarrow \mathcal{E}(-1) \otimes \mathcal{O}_C \longrightarrow 0$$

We have $\mathcal{E}(-1) \otimes \mathcal{O}_C \simeq \mathcal{N}_{C/X}(-2)$. Consider the natural inclusion $\mathcal{N}_{C/X}(-2) \subset \mathcal{N}_{C/\mathbb{P}^4}(-2)$ and apply [Hu, Proposition V.2.1]: $h^1(\mathcal{N}_{C/\mathbb{P}^4}^*(2) \otimes \mathcal{M}) = 0$ for any invertible \mathcal{M} of degree 0. By Serre duality, $h^0(\mathcal{N}_{C/\mathbb{P}^4}(-2)) = 0$. Hence $h^0(\mathcal{N}_{C/X}(-2)) = 0$. The last exact sequence and a) imply that $h^i(\mathcal{I}_C \otimes \mathcal{E}(-1)) = h^i(\mathcal{E}(-1)) = 0$, $i = 0, 1$, hence $h^1(\mathcal{E} \otimes \mathcal{E}(-2)) = h^1(\mathcal{E}(-3))$. By Lemma 2.1, c), and Serre duality, we have $h^2(\mathcal{E}(1)) = h^1(\mathcal{E}(-3)) = 0$, and we are done.

If C is contained in a hyperplane, then $h^0(\mathcal{E}) \neq 0$ by Proposition 2.6, hence $\mathcal{E} \otimes \mathcal{E} = \mathcal{E}nd(\mathcal{E})$ has at least two linearly independent sections: the identity endomorphism, and $s \otimes s$, where s is a non-trivial section of \mathcal{E} .

c) Let C be not contained in a hyperplane. In the exact triple

$$0 \longrightarrow \mathcal{I}_C \otimes \mathcal{E}(1) \longrightarrow \mathcal{E}(1) \longrightarrow \mathcal{N}_{C/X} \longrightarrow 0 \quad (14)$$

we have $h^0(\mathcal{I}_C \otimes \mathcal{E}(1)) = 1$, $h^1(\mathcal{I}_C \otimes \mathcal{E}(1)) = 5$ by b) and (13), and $h^0(\mathcal{E}(1)) = 6$, $h^1(\mathcal{E}(1)) = 0$ by Lemma 2.1, c). Hence $h^0(\mathcal{N}_{C/X}) = 10$. By Riemann–Roch on C , we have $h^1(\mathcal{N}_{C/X}) = 0$.

In the case if $C \subset \mathbb{P}^3$, we can consider C as a curve on a surface, the hyperplane section S of X , as in the proof of Proposition 2.5. For example, assume that the hyperplane section is nonsingular. Then we can represent it as \mathbb{P}^2 with 6 points P_1, \dots, P_6 blown up. Let e_0 be the inverse image of a line in \mathbb{P}^2 , and e_i the exceptional divisors over P_i ($i = 1, \dots, 6$). Then $C \in |3e_0 - e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4}|$ for some subset $\{i_1, i_2, i_3, i_4\} \subset \{1, \dots, 6\}$. The standard exact sequences

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(C) \longrightarrow \mathcal{N}_{C/S} \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{N}_{C/S} \longrightarrow \mathcal{N}_{C/X} \longrightarrow \mathcal{O}_C(1) \longrightarrow 0$$

show that $h^0(\mathcal{N}_{C/X}) = 10$, $h^1(\mathcal{N}_{C/X}) = 0$.

□

Lemma 2.8. *Let C be a smooth elliptic quintic, not contained in a hyperplane. Then any two non-proportional sections of $\mathcal{E}(1)$ define different curves C . Hence the morphism $\mathbb{P}^5 \rightarrow H$ of Corollary 2.4 is injective.*

Proof. By Lemma 2.7, b), $h^0(\mathcal{E} \otimes \mathcal{E}) = 1$. To prove that the section of $\mathcal{E}(1)$ defining C is unique up to a constant factor, it suffices to show that $h^0(\mathcal{I}_C \otimes \mathcal{E}(1)) = 1$. This follows from (13). \square

3. RATIONAL NORMAL QUARTICS

In this section, X is a general cubic threefold. For the future use in Section 4, we prove the irreducibility of the family of rational normal quartics in X .

Theorem 3.1. *There is a unique component of the Hilbert scheme of curves in X whose generic point corresponds to a rational normal quartic.*

We start by several auxiliary results.

Proposition 3.2. *Let X be a nonsingular cubic threefold, and $\mathring{H}_{3,0} = \mathring{H}_{3,0}(X)$ the open subscheme of Hilb_X^{3n+1} parametrizing smooth cubic rational curves lying in nonsingular hyperplane sections of X . Then we have:*

- (i) $\mathring{H}_{3,0}$ is smooth and irreducible of dimension 6.
- (ii) The image of the Abel–Jacobi map $\eta : \mathring{H}_{3,0} \longrightarrow J_1(X)$ is an open subset of a translate Θ_a of the theta divisor of $J_1(X)$.
- (iii) All the scheme-theoretic fibers $\eta^{-1}\eta(s)$ are open subschemes of \mathbb{P}^2 , in particular, they are smooth and irreducible of dimension 2.

Proof. See [We, (6.21)]. We have

$$\eta^{-1}\eta(s) = \{C \in |\mathcal{O}_S(C_s)| : C \text{ is smooth}\}, \quad (15)$$

where $C_s \subset X$ is the curve represented by $s \in \mathring{H}_{3,0}$, $S = \langle C_s \rangle \cap X$, and $\langle C_s \rangle \simeq \mathbb{P}^3$ denotes the linear span of C_s . Thus, the \mathbb{P}^2 from the statement of the proposition is the linear system of C_s on the nonsingular cubic surface S . The image of η can be identified, up to translation, with the image under ψ (see 1.7) of the subset $U = \{(t, t') \in F \times F \mid l_t \cap l_{t'} = \emptyset, \langle l_t, l_{t'} \rangle \cap X \text{ is nonsingular}\}$. The pairs (t, t') are recovered from s as follows: there are exactly 6 lines l_t in S such that $C \in |\mathcal{O}_S(C_s)|$ degenerates into $C^2 + l_t$, where C^2 is a conic. Choose one of them, and let $l_{t'}$ be determined as the residual intersection of the linear span of C^2 with X : $X \cap \langle C^2 \rangle = C^2 \cup l_{t'}$. Then the Abel–Jacobi image of C_s coincides, up to a translation by a constant, with that of $l_t - l_{t'}$. The openness of $\eta(\mathring{H}_{3,0})$ follows from the finiteness of ψ on U [Tyu, §3]. \square

Proposition 3.3. *Let X be a general cubic threefold, $\mathring{H}_{3,0}$ as in Proposition 3.2, and F the Fano surface of X . Let*

$$B = B(X) = \{(s, t) \in \mathring{H}_{3,0} \times F \mid C_s \text{ meets } l_t \text{ transversely at 1 point}\},$$

where C_s (resp. l_t) denotes the rational cubic curve represented by s (resp. the line represented by t). Then B is irreducible of dimension 7.

Proof. Let $b : B \rightarrow \Theta_a$ be defined by $b = \eta \circ p_1$, where Θ_a, η are as in Proposition 3.2, and p_1 is the natural projection from B to $\mathring{H}_{3,0}$. The map b is dominant on every component of B . To see this, notice first that on a general cubic X , the number of lines passing through any point is finite (≤ 6). Indeed, by [Tyu, §1] or [Cl-Gr, Sect. 8], the points $P \in X$ lying on an infinite number of lines are characterized by the property that $T_P X \cap X$ is a cone over a plane cubic curve, and it is easy to check by counting dimensions that there are no such hyperplane sections on a general X .

Hence all the fibers $p_1^{-1}(x)$ are of pure dimension 1. Moreover, any $(x_0, t_0) \in p_1^{-1}(x_0)$ can be obtained as a limit of points $(x, t) \in p_1^{-1}(x)$ for some one-parameter family specializing x into x_0 . Hence B is equidimensional and each one of its components dominates $\mathring{H}_{3,0}$. This implies, by Proposition 3.2, that each component of B dominates Θ_a .

Look now at the fibers of $b : B \rightarrow \Theta_a$. Let $v \in \eta(\mathring{H}_{3,0}) \subset \Theta_a$, $B_v = b^{-1}(v)$, and $f = p_2|_{B_v} : B_v \rightarrow F$, $f : ([C], [l]) \mapsto [l]$. By (15),

$$B_v = \{([C'], [l]) \in \mathring{H}_{3,0} \times F \mid C' \in |\mathcal{O}_S(C)|, l \text{ meets } C' \text{ transversely in 1 point}\}.$$

Hence, if $l \not\subset S$,

$$f^{-1}([l]) = \{\text{open subset of } \mathbb{P}^1 = |\mathcal{I}_{x,S}(C)|\},$$

where $x = l \cap S$. So, all the fibers of $f : B_v \rightarrow F$ are open subsets of \mathbb{P}^1 except for 27 fibers $\simeq \mathbb{P}^2$, corresponding to lines $l_i \subset S$ ($i = 1, \dots, 27$). These exceptional fibers are not irreducible components, as any $l_i \cup C'$ represented by a point of B_v can be deformed into $l \cup C''$ with $l \not\subset S$, $C'' \cap l = \{1 \text{ point}\}$, in using the fact that the family of rational cubics covers S . Hence B_v is irreducible of constant dimension 3, hence B is irreducible, and we are done. \square

Proposition 3.4. *Let X, B be as in Proposition 3.3. Then B can be naturally identified with a subset of $H_{4,0} = \text{Hilb}_X^{4n+1}$. The following assertions are true:*

(i) $H_{4,0}$ is smooth at any point $[Z] = ([C], [l]) \in B$, and

$$\dim T_{[Z]} H_{4,0} = h^0(\mathcal{N}_{Z/X}) = 8, \quad h^1(\mathcal{N}_{Z/X}) = 0.$$

(ii) A curve $Z = C \cup l$ represented by a general point of B can be smoothed into a rational normal quartic inside X .

Proof. (i) We have natural exact sequences

$$0 \longrightarrow \mathcal{N}_{Z/X} \longrightarrow \mathcal{N}_{Z/X}|_C \oplus \mathcal{N}_{Z/X}|_l \longrightarrow \mathcal{N}_{Z/X} \otimes \mathbb{C}(x) \longrightarrow 0, \quad (16)$$

$$0 \longrightarrow \mathcal{N}_{C/X} \longrightarrow \mathcal{N}_{Z/X}|_C \longrightarrow \mathbb{C}(x) \longrightarrow 0, \quad (17)$$

$$0 \longrightarrow \mathcal{N}_{l/X} \longrightarrow \mathcal{N}_{Z/X}|_l \longrightarrow \mathbb{C}(x) \longrightarrow 0, \quad (18)$$

where $x = C \cap l$.

Let $S = \langle C \rangle \cap X$, then $\mathcal{N}_{C/S} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$ via an identification $C \simeq \mathbb{P}^1$. Hence the exact sequence

$$0 \longrightarrow \mathcal{N}_{C/S} \longrightarrow \mathcal{N}_{C/X} \longrightarrow \mathcal{N}_{S/X}|_C \longrightarrow 0$$

yields

$$\mathcal{N}_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \text{ or } \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3). \quad (19)$$

By (4), (19), (17) and (18), $h^1(\mathcal{N}_{C/X}) = h^1(\mathcal{N}_{l/X}) = 0$, $h^0(\mathcal{N}_{Z/X}|_C) = 7$, $h^0(\mathcal{N}_{Z/X}|_l) = 3$. The map $H^0(\mathcal{N}_{Z/X}|_C \oplus \mathcal{N}_{Z/X}|_l) \longrightarrow H^0(\mathcal{N}_{Z/X} \otimes \mathbb{C}(x))$ arising from (16) is surjective, because $\mathcal{N}_{Z/X}$ is locally free and $\mathcal{N}_{Z/X}|_C \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$ or $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(4)$ is generated by global sections. Hence $h^0(\mathcal{N}_{Z/X}) = 8$, $h^1(\mathcal{N}_{Z/X}) = 0$.

(ii) We will use the following particular case of [H-H, Theorem 4.1] (though it is formulated in [H-H] for nodal curves in \mathbb{P}^3 , its proof remains valid with \mathbb{P}^3 replaced by any nonsingular projective variety).

Lemma 3.5. *Let X be a nonsingular projective variety, C_1, C_2 two smooth curves in X meeting transversely at one point x . Assume that $H^1(C_i, \text{elm}_x^- \mathcal{N}_{C_i/X}) = H^1(C_{3-i}, \mathcal{N}_{C_{3-i}/X}) = 0$ for at least one $i = 1$ or 2 . Then $H^1(\mathcal{N}_{C_1 \cup C_2/X}) = 0$ and $C_1 \cup C_2$ is strongly smoothable in X .*

Here elm_x^- denotes the negative elementary transformation in x (see loc. cit., Sect. 2 for a definition).

From (19), we obtain

$$\text{elm}_x^- \mathcal{N}_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \text{ or } \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(3).$$

Hence $h^1(\text{elm}_x^- \mathcal{N}_{C/X}) = 0$. We have already seen that $h^1(\mathcal{N}_{l/X}) = 0$, hence the lemma implies the result. \square

Proof of Theorem 3.1. Let $H'_{4,0}(X)$ be the component of $H_{4,0}(X)$ containing $B(X)$, where $B(X)$ is defined in Proposition 3.3. It is unique, because $H_{4,0}(X)$ is smooth and of dimension 8 at the points of $B(X)$, and thus we have $\dim H'_{4,0}(X) = 8$.

Let $\overset{\circ}{H}_{4,0} \subset \text{Hilb}_{\mathbb{P}^4}^{4n+1}$ be the subscheme parametrizing rational normal quartics; it is a smooth homogeneous manifold of dimension 21: $\overset{\circ}{H}_{4,0} \simeq PGL_5(\mathbb{C})/PGL_2(\mathbb{C})$. Let

$$I = \{(X, C) \in |\mathcal{O}_{P^4}(3)| \times \overset{\circ}{H}_{4,0} \mid C \subset X\}$$

be the incidence variety, p_1, p_2 its projections to the factors of $|\mathcal{O}_{P^4}(3)| \times \overset{\circ}{H}_{4,0}$. Then $p_2^{-1}([C]) = |\mathcal{I}_{C,P^4}(3)| \simeq \mathbb{P}^{21}$ for any C . Hence I is irreducible of dimension 42. We have seen that there is an irreducible component of a general fiber of $p_1 : I \rightarrow \mathbb{P}^{34} = |\mathcal{O}_{P^4}(3)|$ of dimension 8, hence p_1 is dominant. Moreover, the irreducibility of I implies that the action of the monodromy group on the irreducible components of a general fiber is transitive. As we can distinguish one particular component $H'_{4,0}(X)$, invariant under monodromy, the general fiber of p_1 is irreducible. \square

Remark 3.6. The basic idea of our proof of the irreducibility of the family of rational normal quartics in X is degenerating a quartic into a reducible curve, which reduces the problem to questions on lower degree curves. This step could be done using the “bend and break technique” [Mo], [Ko, II.5], which gives a deformation Γ' of Γ whose associated 1-cycle is of the form $\sum a_i \Gamma_i$ with $\sum a_i \geq 2$ (see [Ko, Corollary II.5.6]). In order to fulfil the hypotheses necessary for the application of this technique, one has to verify that there are enough rational quartics in X passing through two general points. Each component of the Hilbert scheme of curves in X whose generic point is a smooth rational normal quartic is of dimension 8. Hence, one can expect that in each component, there is a 4-dimensional family of curves passing through two general points $P_1, P_2 \in X$. To verify that this is really the case, the standard argument with incidence varieties can be used (compare to Lemma 4.3). One has to show that there is a component of dimension 4 in the subset of the Hilbert scheme of rational quartics passing through two general points in a *particular* cubic threefold. One can easily check that this is so on a cone over a smooth cubic surface $Y \subset \mathbb{P}^3$.

4. NORMAL ELLIPTIC QUINTICS

4.1. Let H_0 be the component of $\text{Hilb}_{\mathbb{P}^4}^{5n}$ whose generic point corresponds to a smooth normal elliptic quintic in \mathbb{P}^4 . H or $H(X)$ will

denote, as before, the union of components of Hilb_X^{5n} having smooth normal elliptic quintics as their generic points (hence contained in H_0). It is well-known that H_0 is unique, of dimension 25, and is generically parametrized by 24 parameters corresponding to the action of $PGL_5(\mathbb{C})$, and by the modulus of the elliptic curve (see [Hu], or [H-H, Proposition 6.1] for the irreducibility of Hilbert schemes of curves in a more general context).

Lemma 4.2. *Through any smooth elliptic normal quintic in \mathbb{P}^4 passes a 19-dimensional linear system of cubic hypersurfaces, and its general member is nonsingular.*

Proof. The dimension calculation follows from (8) for $k = 3$. By [Hu, Theorem IV.1.3] or by [Mum-2, Theorem 10], C is a scheme-theoretic intersection of quadrics passing through C . Hence this is also true for cubics, and by Bertini's Theorem, the general cubic through C is nonsingular. \square

Lemma 4.3. *Let $I \subset H_0 \times \mathbb{P}^{34}$ be the incidence variety parametrizing the pairs (C, X) consisting of a smooth normal elliptic quintic C and a cubic hypersurface X containing C . Then I is irreducible of dimension 44, and its projection to \mathbb{P}^{34} is dominant. The fiber of this projection has dimension 10 and is smooth at any point $[C]$ representing a normal elliptic quintic C in a nonsingular cubic threefold.*

Proof. The irreducibility of I follows from [Sh, Theorem 1.6.12]. Now, take any nonsingular cubic X_0 containing a smooth normal elliptic quintic C_0 , which is possible by Lemma 4.2. Then, by Lemma 2.7, c), $h^0(\mathcal{N}_{C_0/X_0}) = 10$, $h^1(\mathcal{N}_{C_0/X_0}) = 0$, hence $H(X_0)$ is smooth of dimension 10 at $[C_0]$ (see [Gro]). \square

Sofar, we have proved:

Proposition 4.4. *Let X be a general cubic threefold. Then $H = H(X)$ is the union of finitely many irreducible components of dimension 10. The generic point of any of these components is smooth and corresponds to a smooth normal elliptic quintic of \mathbb{P}^4 contained in X . For any such quintic C , the point $[C] \in H$ is smooth.*

Theorem 4.5. *Let X be a general cubic threefold, and $H = H(X)$. Then H is irreducible of dimension 10.*

Proof. Choose any component H' of H . By Proposition 4.4, there is a smooth normal quintic $C \subset \mathbb{P}^4$ corresponding to a point of H' . Choose a linear series g_2^1 on C , and construct the rational cubic scroll Σ as the

union of lines joining pairs of points in g_2^1 . One can easily verify that Σ is the Hirzebruch surface \mathbb{F}_1 embedded by the linear system of degree 3, and its intersection with X is C plus a rational normal quartic Γ or, possibly, a degenerate curve of the linear system of rational normal quartics in Σ . Such curves Γ meet the (-1) -curve L of Σ in two points. Notice, that the family of scrolls Σ obtained by this construction is connected of dimension 1, parametrized by the points of $\text{Pic}^2(C) \simeq C$.

As a next step, we want to invert this construction and to recover C from Γ . It is convenient to reduce the problem to the case when Γ is smooth (see Remark 4.8, giving an idea, how to avoid this reduction). The singular members of $|\Gamma|$ on Σ are of the form $\Gamma' + \sum a_i F_i$, with $1 \leq \sum a_i \leq 3$, where Γ' (resp. F_i) is a section (resp. a fiber) of the scroll Σ , and F_i are chords of C lying in X . So, if C has only finitely many chords in X , then we can always choose a g_2^1 on C in such a way that it contains no chords, and then Γ will be nonsingular. The finiteness of the number of chords is guaranteed by the following lemma.

Lemma 4.6. *A general elliptic quintic C in any component of $H(X)$ for a general cubic threefold X has only finitely many chords in X .*

Postponing the proof of the lemma, let us show now that given any normal rational quartic $\Gamma \in \mathbb{P}^4$ contained in X , we can find a cubic scroll Σ such that $\Sigma \cap X = \Gamma \cup C$ for some quintic C . It can be constructed geometrically as follows. Take any chord L of Γ , meeting Γ in two distinct points P_1, P_2 . Choose arbitrarily two points $P_3 \in \Gamma$ and $P'_3 \in L$, different from P_1, P_2 . Connect each point $M \in \Gamma$ by a line with $M' \in L$, where M' is defined by the equality of the cross-ratios $[P_1, P_2, P_3, M] = [P_1, P_2, P'_3, M']$. The union of these lines is a cubic scroll Σ having L as (-1) -curve. In the case when $P_1 = P_2 = P$, L is tangent to Γ at P , and the scroll is determined by an isomorphism $\phi : \Gamma \rightarrow L$ such that $\phi(P) = P$, $d_P \phi = \text{id}_{T_P L}$ (with a natural identification of $T_P L$ and $T_P \Gamma$). The residual curve in the intersection with X yields a quintic C . We see that the family of the scrolls Σ constructed from a fixed quartic curve is irreducible and has dimension 3. Indeed, it is parametrized by a 3-dimensional variety fibered over $\text{Sym}^2 \Gamma$ with fiber over a pair of points $(P_1, P_2) \in \text{Sym}^2 \Gamma$ equal to $\text{Span}(P_1, P_2) \setminus \{P_1, P_2\}$, if $P_1 \neq P_2$, and to the curve of isomorphisms $\Gamma \rightarrow L$ fixing $P = \Gamma \cap L$ and $T_P L$, if $P_1 = P_2 = P$.

So, our construction yields a 3-dimensional irreducible family of scrolls whose residual intersection with X defines a quintic curve. Taking into account Proposition 4.4, we obtain the following statement:

Proposition 4.7. *Let U be the open set of H parametrizing general (in the sense of Lemma 4.6) smooth normal elliptic quintics of \mathbb{P}^4 lying*

on a general cubic threefold X , and V the open subset of the Hilbert scheme of quartic curves in X parametrizing all the curves which are residual intersections with X of the cubic scrolls through quintics C with $[C] \in U$. Then the above construction establishes a correspondence $Z \subset V \times U$ with irreducible fibers of dimension 3 over V and of dimension 1 over U . In particular, U and V have equal numbers of irreducible components.

The proof of Theorem 4.5 is ended by the application of Theorem 3.1. \square

Proof of Lemma 4.6. Take C smooth and projectively normal in any component $H'_0 \subset H(X_0)$ of a given nonsingular cubic threefold X_0 . Let $\pi_P : C \rightarrow C_1$ be a general projection from a point $P \in \mathbb{P}^4 \setminus X_0$ to $\mathbb{P}^3 \subset \mathbb{P}^4$. Then C_1 is an elliptic quintic in \mathbb{P}^3 . There exists a nonsingular cubic surface Y containing C_1 ; the proof of this assertion is similar to that of Lemma 4.2. Let X_1 be the cone over Y with vertex P . Then X_1 is also a cubic threefold containing C . Every chord of C lying in X_1 is projected by π into a chord of C_1 lying in Y . As the number of lines in a nonsingular cubic surface is equal to 27, the number of chords of C in X_1 is finite. Hence it is also finite in a general member X_λ of the pencil of cubics (X_0, X_1) . As we can start with C in any component of $H(X_0)$, we see that a general C in any component has only finitely many chords in X_t for a general deformation X_t of X_0 , and we are done. \square

Remark 4.8. The construction of the cubic scroll through a rational quartic extends also to the case when Γ is a singular member of the corresponding linear system on some scroll $\Sigma \simeq \mathbb{F}_1$. Indeed, this linear system is $|L + 3F|$, where F denotes the fiber of the ruled surface, and the only possible degenerate members are of one of the following types: a) $L + F_1 + F_2 + F_3$, where F_i are lines on X meeting C in two points (chords), or tangent to C ; b) $C_2 + F_1 + F_2$, where C_2 is a smooth conic; c) $C_3 + F_1$, where C_3 is a twisted cubic in some \mathbb{P}^3 . Treat, for example, the case a); the arguments are similar in the remaining cases. Let $P_i = F_i \cap L$, $i = 1, 2, 3$. The three lines, whether they are distinct or not (in the last case we consider them as a non-reduced subscheme of \mathbb{P}^4), do not lie in one hyperplane. Otherwise, this hyperplane would have the intersection number at least 6 with C , and hence C would lie in the hyperplane, which contradicts our assumptions. So, through three points $Q_i \in F_i$ (possibly infinitesimal, in the non-reduced case), different from P_i , passes a unique 2-plane Π . Choosing in this plane a conic B through the Q_i , we can join the points of L, B with the same

cross-ratios with respect to $P_i = L \cap F_i, Q_i$. This construction giving a scroll $\Sigma \supset \Gamma$ works in the reduced case, and can be extended to the non-reduced one by an obvious passage to the limit.

Remark 4.9. We saw in the proof of Lemma 4.3, c) that the smooth normal elliptic quintics are represented by smooth points of the Hilbert scheme of curves. In fact, smooth non-normal quintics (that is, lying in a hyperplane), are represented also by smooth points in H , though they may be rather “singular” in other situations: they define unstable vector bundles of rank 2, and their images under the Abel–Jacobi map coincide with those of pairs of lines $l_1 + l_2$ ($l_i \subset X$).

Indeed, $h^0(\mathcal{N}_{C/X}) = 10, h^1(\mathcal{N}_{C/X}) = 0$ by Lemma 2.7, so $[C]_X$ is a smooth point. It belongs to the component H of Hilb_X^{5n} , because the family of elliptic quintics in hyperplane sections of X is 9-dimensional, so that C has deformations to smooth quintics not contained in a hyperplane.

5. FACTORIZATION OF THE ABEL–JACOBI MAP THROUGH MODULI OF VECTOR BUNDLES

Let X be a general cubic threefold. Let $M_X(2; 0, 2)$ be the Gieseker–Maruyama moduli space [Ma], [Sim] of semistable (with respect to $\mathcal{O}_X(1)$) rank 2 torsion free sheaves on X with Chern classes $c_1 = 0$ and $c_2 = 2[l]$, where $[l]$ is the class of a line modulo numerical equivalence. Define the Zariski open subset M_0 in it as follows:

$$M_0 = \{[\mathcal{E}] \in M_X(2; 0, 2) \mid (i) \mathcal{E} \text{ is stable and locally free;} \\ (ii) h^1(\mathcal{E}(-1)) = h^1(\mathcal{E}(1)) = h^2(\mathcal{E}(1)) = h^2(\mathcal{E} \otimes \mathcal{E}) = 0\}. \quad (20)$$

Lemma 5.1. *Let \mathcal{E} be a sheaf on X with $[\mathcal{E}] \in M_0$. Then we have:*

- a) $h^0(\mathcal{E}(1)) = 6, h^i(\mathcal{E}(1)) = 0$ for $i > 0$.
- b) *The scheme of zeros C of any section $s \neq 0$ of $\mathcal{E}(1)$ is of pure dimension 1 and is not contained in a hyperplane.*
- c) $h^i(\mathcal{E}(-1)) = 0 \quad \forall i \in \mathbb{Z}$.
- d) $h^0(\mathcal{E} \otimes \mathcal{E}) = 1, h^1(\mathcal{E} \otimes \mathcal{E}) = 5, h^2(\mathcal{E} \otimes \mathcal{E}) = h^3(\mathcal{E} \otimes \mathcal{E}) = 0$.
- e) $h^0(\mathcal{N}_{C/X}) = 10, h^1(\mathcal{N}_{C/X}) = 0$.

Proof. This follows immediately from the definition of M_0 , the Serre duality, exact sequences (12), (14), and from Riemann–Roch–Hirzebruch for $\mathcal{E}(1), \mathcal{E} \otimes \mathcal{E}$. \square

Define now the locally closed subscheme $\mathcal{H}_0 \subset \text{Hilb}_X^{5n}$:

$$\mathcal{H}_0 = \left\{ [C] \in \text{Hilb}_X^{5n} \mid \begin{array}{l} \text{(i) } C \text{ is a locally complete in-} \\ \text{tersection of pure dimension 1, (ii) } h^1(\mathcal{I}_C) = \\ h^1(\mathcal{I}_C(1)) = 0 \text{ (hence } h^0(\mathcal{O}_C) = 1), \text{ (iii) } \\ h^1(\mathcal{I}_C(2)) = h^2(\mathcal{I}_C(2)) = 0 \text{ (hence } h^0(\mathcal{I}_C(2)) = \\ 5), \text{ (iv) } \omega_C \simeq \mathcal{O}_C \end{array} \right\} \quad (21)$$

Lemma 5.2 (Relative Serre's construction). *There is a well-defined morphism $\phi : \mathcal{H}_0 \rightarrow M_0$, $[C] \mapsto [\mathcal{E}]$, where \mathcal{E} is defined by the exact triple (γ) , determined by a non-zero extension class in $H^0(\omega_C) = \text{Ext}^1(\mathcal{I}_C(2), \mathcal{O}_X)$.*

Proof. The proof consists in an obvious relativization of Serre's construction of Sect. 2 (see Corollary 2.2). □

Lemma 5.3. *\mathcal{H}_0 is isomorphic to the projectivization of a rank 6 vector bundle locally in the étale topology over M_0 .*

Proof. According to Simpson [Sim, Theorem 1.21], each point of M_0 has an étale neighborhood $T \rightarrow U \subset M_0$, such that there exists a Poincaré vector bundle \mathcal{E} on $X \times T$. Let $\mathcal{G} = \text{pr}_{2*}(\mathcal{E} \otimes \mathcal{O}_X(1))$, a locally free sheaf of rank 6 on T . Let \mathcal{C} be the universal family of curves over \mathcal{H}_0 , and $\mathcal{C}_U \rightarrow \mathcal{H}_0^U$ its restriction over U .

Given a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & \mathcal{H}_0^U \\ \beta \downarrow & & \downarrow \phi \\ T & \xrightarrow{\lambda} & U \end{array}$$

we will show that there exists a unique morphism $\gamma : Y \rightarrow \mathbb{P}(\mathcal{G})$ such that $q \circ \gamma = \alpha$, $p \circ \gamma = \beta$, where $p : \mathbb{P}(\mathcal{G}) \rightarrow T$ is the structure morphism and $q : \mathbb{P}(\mathcal{G}) \rightarrow \mathcal{H}_0$ is the natural map sending the proportionality class of a section s_t of $\mathcal{E}_t(1)$ on $X \times \{t\}$ to its scheme of zeros $Z_X(s_t)$. Then, by the universal property of the Cartesian product, $\mathbb{P}(\mathcal{G}) = \mathcal{H}_0^U \times_U T$.

Lift \mathcal{C}_U to the family $p_2 : \mathcal{C}_Y \rightarrow \mathcal{H}_0^U = Y \times_U \mathcal{H}_0$. By conditions (ii), (iv) of (21) and Base Change, $p_{2*}\omega_{\mathcal{C}_Y/Y}$ is invertible, so that there exists an open covering $Y = \cup_{i \in I} Y_i$ and non-vanishing sections $\xi_j \in \Gamma(Y_j, p_{2*}\omega_{\mathcal{C}_Y/Y})$. They determine extensions

$$0 \rightarrow \mathcal{O}_{X \times Y_j} \xrightarrow{\mu_j} \tilde{\mathcal{E}}_j(1) \rightarrow \mathcal{I}_{\mathcal{C}_j}(2) \rightarrow 0,$$

where $\mathcal{C}_j = \mathcal{C}_Y|_{Y_j}$, such that $[\tilde{\mathcal{E}}_j|_{X \times \{y\}}] = \lambda \circ \beta(y) \forall y \in Y_j$.

Hence, denoting $\beta_j = \beta|_{Y_j}$, we get by the universal property of the Poincaré bundle the isomorphisms $\tilde{\mathcal{E}}_j \simeq (1 \times \beta_j)^* \mathcal{E} \otimes \mathcal{M}_j$ for some invertible sheafs \mathcal{M}_j on Y_j . Applying p_{2*} to μ_j , we obtain the monomorphism $p_{2*}\mu_j : \mathcal{O}_{Y_j} \hookrightarrow \beta_j^* \mathcal{G} \otimes \mathcal{M}_j$, or, by duality, an epimorphism $\epsilon_j : \beta_j^* \mathcal{G}^* \rightarrow \mathcal{M}_j$. By the universal property of the functor **Proj** [Ha, Exercise II.7.8], there exists a unique morphism $\gamma_j : Y_j \rightarrow \mathbb{P}(\mathcal{G})$ such that $\epsilon_j = \gamma_j^* \epsilon$, where $\epsilon : p^* \mathcal{G}^* \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{G})/T}(1)$ is the canonical epimorphism. These morphisms agree on the intersections $Y_i \cap Y_k$ and thus define a morphism $\gamma : Y \rightarrow \mathbb{P}(\mathcal{G})$. By construction, $q \circ \gamma = \alpha$, $p \circ \gamma = \beta$. \square

Corollary 5.4. *The morphism $\phi : \mathcal{H}_0 \rightarrow M_0$ defined in Lemma 5.2 is smooth, projective, and all its fibers are 5-dimensional projective spaces.*

Proof. According to Lemma 5.3, $\phi : \mathcal{H}_0 \rightarrow M_0$ is identified, locally in the étale topology, with the natural projection $\mathbb{P}^5 \times U \rightarrow U$. \square

Corollary 5.5. *\mathcal{H}_0, M_0 are smooth of dimensions 10, resp. 5; moreover, $\text{Hilb}_X^{5n}, M_X(2; 0, 2)$ are smooth at the points of \mathcal{H}_0 , resp. M_0 .*

Proof. This follows from Lemma 5.1 d), e). \square

We do not treat the question of irreducibility of \mathcal{H}_0 and M_0 . Let \mathcal{H}'_0 be the irreducible component of \mathcal{H}_0 containing the open subset

$$\mathcal{H}^* = \{[C] \in \mathcal{H}_0 \mid C \text{ is a smooth projectively normal elliptic quintic}\}.$$

\mathcal{H}^* is irreducible by Theorem 4.5, and we proved in Section 2 that $\mathcal{H}^* \subset \mathcal{H}_0$. Define also $M = \phi(\mathcal{H}^*)$, $\mathcal{H} = \phi^{-1}(M) \cap \mathcal{H}'_0$. In the sequel, we will denote the restriction $\phi|_{\mathcal{H}}$ by the same symbol ϕ . By Corollaries 5.4, 5.5, M, \mathcal{H} are open and smooth.

Theorem 5.6. *Let $\mathcal{H}^* \subset \mathcal{H} \subset \text{Hilb}_X^{5n}, M \subset M_X(2; 0, 2)$, $\phi : \mathcal{H} \rightarrow M$ be defined as above. The following assertions are true:*

(i) *For every choice of a reference point $[C_0] \in \mathcal{H}$, the Abel–Jacobi map $[C] \mapsto [C - C_0]$ defines a morphism $\Phi : \mathcal{H} \rightarrow J_1(X)$. Fix some $[C_0]$ and the corresponding morphism.*

(ii) *Φ is smooth, and every fiber of Φ is a disjoint union of 5-dimensional projective spaces.*

(iii) *There exists a quasi-finite étale morphism $\Psi : M \rightarrow J_1(X)$ such that $\Phi = \Psi \circ \phi$.*

Proof. (i) \mathcal{H} is smooth and is a base of a flat family of curves, hence, by 1.3, the Abel–Jacobi map $\Phi : \mathcal{H} \rightarrow J_1(X)$ is well-defined as an analytic map. It is in fact algebraic. Indeed, \mathcal{H} is a subvariety of the Chow variety of 1-cycles in X (see [Ko, Theorem I.6.3]), and we can resolve the singularities of the closure of \mathcal{H} in the Chow variety. Then

Φ extends to this resolution as an analytic map. It will be projective by GAGA principle, as the Chow variety is projective.

(ii) First, we will verify that Φ is smooth at any point of \mathcal{H}^* . This follows from the computation of the differential of Φ by the technique of TBS (see 1.9). Apply it to $X \subset W = \mathbb{P}^4$. The map R of (5) is an isomorphism $H^0(X, \mathcal{O}_X(1)) \xrightarrow{\sim} H^1(X, \Omega_X^2)$ by [Cl-Gr, (12.7)]. We have to verify that the map ψ_Z^* of (5) is injective for $Z = C$ with $[C] \in \mathcal{H}^*$. It suffices to show the injectivity of r_C, β_C . The kernel of r_C lies in $H^0(X, \mathcal{N}_{X/\mathbb{P}^4} \otimes \omega_X \otimes \mathcal{I}_{C/X}) = H^0(\mathcal{I}_C(1)) = 0$ by Lemma 2.1 b). By (6), β_C is a part of the exact sequence

$$\begin{aligned} \dots \longrightarrow H^0(\Omega_X^3 \otimes \mathcal{N}_{C/\mathbb{P}^4}) \longrightarrow H^0(\Omega_X^3 \otimes \mathcal{N}_{X/\mathbb{P}^4}|_C) \xrightarrow{\beta_C} \\ H^1(\Omega_X^3 \otimes \mathcal{N}_{C/X}) = H^0(\mathcal{N}_{C/X})^*. \end{aligned}$$

By [Hu, V.2.1], $H^0(\Omega_X^3 \otimes \mathcal{N}_{C/\mathbb{P}^4}) = 0$. Hence $\beta_C : H^0(\mathcal{O}_C(1)) \longrightarrow H^0(\mathcal{N}_{C/X})^*$ is injective. Hence Φ is of maximal rank, equal to 5, at $[C]$.

Thus, every $z \in \mathcal{H}^*$ is contained in a 5-dimensional component \mathcal{H}_z of the fiber $\Phi^{-1}\Phi(z)$, nonsingular at z . We know also that every projective space contained in \mathcal{H} is contracted by Φ , because the Abel–Jacobi map contracts the rational equivalence. Hence $\mathcal{H}_z = \phi^{-1}\phi(z)$, the fiber of ϕ .

More generally, every fiber $\Phi^{-1}(w)$ is the union of projective spaces $\phi^{-1}\phi(z) \simeq \mathbb{P}^5$ over all $z \in \Phi^{-1}(w)$, and for $z \neq z' \in \Phi^{-1}(w)$ we have: either $\phi^{-1}\phi(z) = \phi^{-1}\phi(z')$, or $\phi^{-1}\phi(z) \cap \phi^{-1}\phi(z') = \emptyset$. By construction, every fiber \mathbb{P}^5 of ϕ contains a point of \mathcal{H}^* , hence $\Phi^{-1}(w)$, if non-empty, is a disjoint finite union of irreducible components of the form $\phi^{-1}\phi(z) \simeq \mathbb{P}^5$ with $z \in \Phi^{-1}(w)$, and there are no multiple components among them, since the fiber should be scheme-theoretically smooth at points of \mathcal{H}^* . Moreover, the fibers of Φ are locally complete intersections in \mathcal{H} , because they are of pure codimension 5 and $J_1(X)$ is smooth of dimension 5, so, they have no embedded points. Hence all the components of the fibers of Φ are smooth subvarieties of \mathcal{H} , isomorphic to \mathbb{P}^5 , with their reduced structure.

(iii) By (ii), the map Ψ is well-defined set-theoretically: we saw that M parametrizes the irreducible components (isomorphic to \mathbb{P}^5) of the fibers of Φ . By Corollary 5.4, $\phi_*\mathcal{O}_{\mathcal{H}} = \mathcal{O}_M$, and ϕ is open. For every open $U \subset J_1(X)$, the inverse image $\Psi^{-1}(U)$ under our set-theoretical map is open, because $\Psi^{-1}(U) = \phi(\Phi^{-1}(U))$, and $\Psi^*(g)$ is a regular function on $\Psi^{-1}(U)$ for every $g \in \mathcal{O}_{J_1(X)}(U)$. Thus Ψ is Zariski continuous and lifts regular functions on a Zariski open subset $U \subset J_1(X)$ to those on $\Psi^{-1}(U)$. Hence Ψ is a morphism. We have $d\Phi = d\Psi \circ d\phi$ and $d\Phi$ is surjective. Hence $d\Psi$ is surjective, hence it is an isomorphism and Ψ is étale. \square

Leaving aside the hard question on the complete description of the boundary of \mathcal{H} in Hilb_X^{5n} and of its image in $J_1(X)$, we can easily determine the image of one of the components of the boundary, namely, the one whose general point represents a projectively non-normal elliptic quintic. Let H_{ns} be the nonsingular locus of H , $B^* \subset H$ the subset of classes of smooth elliptic quintics contained in nonsingular hyperplane sections of X , and B the closure of B^* in H . By Remark 4.9, $B^* \subset H_{\text{ns}}$ and B are of pure dimension 9. According to 1.3, the Abel–Jacobi map Φ of Theorem 1.3 extends to a morphism, well defined on H_{ns} , hence on B^* and on the desingularization of B .

Proposition 5.7. *Let $\tilde{\Phi} : H_{\text{ns}} \rightarrow J_1(X)$ be the Abel–Jacobi map defined above. The following properties are verified:*

- (i) *B is an irreducible divisor in H .*
- (ii) *The image $\tilde{\Phi}(B^*)$ of B^* in $J_1(X)$ is dense in the translate Δ_a by an element $a \in J_1(X)$ of the divisor $\Delta = \tilde{\psi}(F \times F)$ defined in 1.8.*
- (iii) *Let $\eta = \Phi|_{B^*} : B^* \rightarrow \Delta_a$ be the restriction of $\tilde{\Phi}$. Then the general fiber of η is an open subset of the 5-dimensional projective space \mathbb{P}^5 , realized as the subset of nonsingular curves in a linear system on a cubic surface.*
- (iv) *$\tilde{\Phi}$ is ramified along B . More exactly, $\text{corank } d\tilde{\Phi} = 1$ at every point of B^* .*

Proof. By Proposition 2.5, a curve C with $[C] \in B^*$ determines a unique unordered pair of disjoint lines l_1, l_2 on the nonsingular cubic surface $S = X \cap \langle C \rangle$ in such a way that $C \in |h + l_1 + l_2| \simeq \mathbb{P}^5$, where $\langle C \rangle$ is the linear span of C in \mathbb{P}^4 , and h is the hyperplane section of S . The pairs of disjoint lines are parametrized by an irreducible 4-fold $\text{Sym}^2 F \setminus (\text{incidence divisor})$, and the pairs whose span defines a nonsingular hyperplane section form an open subset U in it. Hence B^* is irreducible, and its image in $J_1(X)$ coincides with $\tilde{\psi}(U)$ modulo a translation depending on the choice of reference points for $\tilde{\psi}, \Phi$. By Beauville’s remark in 1.8, the degree of $\tilde{\psi}$ over its image is 2, so there is a unique unordered pair l_1, l_2 over the generic point of Δ_a . This proves (i)–(iii). The assertion (iv) follows from the technique of TBS: as in the proof of Theorem 5.6, (ii), for $[C] \in B^*$, we find that β_C is injective, but $\ker r_C = H^0(\mathcal{I}_C(1))$ is of dimension 1, which implies the result. \square

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